A syntax for cubical type theory

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Agda Ideas' Meeting XIX
May 23, 2014

Problem

- Goal: a type theory with the property:
  if two objects are indistinguishable by observation, they are equal
- A candidate: homotopy type theory
  - Equality is defined by an inductive type with the J eliminator
  - Addition of the univalence axiom ("isomorphic types are equal")
  - We don't know how to run progams involving this axiom

Plan

- Homotopy type theory teaches us that equality can be described individually for each type former, eg.:

$$
\begin{array}{lrcl}
\text{pairs:} & ((a, b) =_{A\times B} (a', b')) & \simeq & (a =_A a' \times b =_B b') \\
\text{functions:} & (f =_{A\to B} g) & \simeq & (\Pi(x : A).f\,x =_B g\,x) \\
\text{natural numbers:} & (\text{zero} =_{\mathbb{N}} \text{zero}) & \simeq & 1 \\
 & (\text{zero} =_{\mathbb{N}} \text{suc}\,m) & \simeq & 0 \\
 & (\text{suc}\,m =_{\mathbb{N}} \text{zero}) & \simeq & 0 \\
 & (\text{suc}\,m =_{\mathbb{N}} \text{suc}\,n) & \simeq & (m =_{\mathbb{N}} n)
\end{array}
$$

- Let's define equality separately for each type former, as above!

Inspiration and structure of talk

This work is based on the following papers:

- Bernardy, Moulin: A computational interpretation of parametricity, 2012
- Bezem, Coquand, Huber: A cubical set model of type theory, 2013

Table of contents:

Basic setup

- Type theory with explicit substitutions, without the identity type
- Judgement types:

$$\vdash \Gamma$$
$$\Gamma \vdash u : A$$
$$\rho : \Gamma \Rightarrow \Delta$$
$$\Gamma \vdash u \equiv v : A$$
$$\rho \equiv \delta : \Gamma \Rightarrow \Delta$$

- Applying substitutions:

$$\frac{\Gamma \vdash a : A \quad \rho : \Delta \Rightarrow \Gamma}{\Delta \vdash a[\rho] : A[\rho]} \qquad \frac{\Gamma \vdash A : \mathsf{U} \quad \rho : \Delta \Rightarrow \Gamma \quad \Delta \vdash t : A[\rho]}{(\rho, x \mapsto t) : \Delta \Rightarrow \Gamma, x : A}$$

Heterogeneous equality (i)

- For elements of a Σ-type, the second equality depends on the first:

$$((a, b) =_{\Sigma(x:A).B\,x} (a', b')) \simeq (\Sigma(r : a =_A a').\mathsf{transport}_B\, r\, b =_{B\,a'} b')$$

- To model this, we will a heterogeneous equality: a binary logical relation.

Heterogeneous equality (ii)

- The heterogeneous equality relation:

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^{=} \vdash \sim_A : A[0_\Gamma] \to A[1_\Gamma] \to \mathsf{U}}$$

- $\Gamma^{=}$ is the context containing two copies of the context Γ and proofs that they are related.

$$\begin{aligned} \emptyset^{=} &\equiv \emptyset \\ (\Gamma, x : A)^{=} &\equiv \Gamma^{=}, x_0 : A[0_\Gamma], x_1 : A[1_\Gamma], \bar{x} : x_0 \sim_A x_1 \end{aligned}$$

- 0, 1 project out the corresponding components.

$$\begin{aligned} i_\emptyset &\equiv () &&: \emptyset \Rightarrow \emptyset \\ i_{\Gamma,A} &\equiv (i_\Gamma, x \mapsto x_i) &&: (\Gamma, x : A)^{=} \Rightarrow \Gamma, x : A \end{aligned}$$

$\sim$ on different type formers

Given $\Gamma \vdash A$, $\Gamma, x : A \vdash B$, we previously had:

$$\frac{\Gamma \vdash (a, b) : \Sigma(x : A).B \quad \Gamma \vdash (a', b') : \Sigma(x : A).B}{\Gamma \vdash ((a, b) =_{\Sigma(x:A).B} (a', b')) \simeq (\Sigma(r : a =_A a').\mathsf{transport}_{\lambda x.B[x]}\, r\, b =_{B[a']} b')}$$

Now we have:

$$\frac{\Gamma.A \vdash B : \mathsf{U} \quad \Gamma^{=} \vdash (a, b) : (\Sigma A\, B)[0] \quad \Gamma^{=} \vdash (a', b') : (\Sigma A\, B)[1]}{\begin{array}{l}\Gamma^{=} \vdash (a, b) \sim_{\Sigma(x:A).B} (a', b') \equiv \Sigma(r : a \sim_A a'). \\ \qquad\qquad\qquad\qquad b \sim_B [x_0 \mapsto a, x_1 \mapsto a', \bar{x} \mapsto r]\, b' : \mathsf{U}\end{array}}$$

For Π types:

$$\frac{\Gamma.A \vdash B : \mathsf{U} \quad \Gamma^{=} \vdash f_0 : (\Pi A\, B)[0] \quad \Gamma^{=} \vdash f_1 : (\Pi A\, B)[1]}{\Gamma^{=} \vdash f_0 \sim_{\Pi A\, B} f_1 \equiv \Pi(x_0 : A[0], x_1 : A[1], \bar{x} : x_0 \sim_A x_1).f_0\, x_0 \sim_B f_1\, x_1 : \mathsf{U}}$$

For the universe (we will replace this later):

$$A \sim_{\mathsf{U}} B \equiv A \to B \to \mathsf{U}$$

Every term is a congruence

We validate the rule

$$\frac{\Gamma \vdash u : A}{\Gamma^= \vdash u^\sim : u[0_\Gamma] \sim_A u[1_\Gamma]}$$

for each term former. This corresponds to showing that every term is parametric, eg.:

$$\frac{\Gamma.x : A \vdash b : B}{\Gamma^= \vdash (\lambda x.b)^\sim \equiv \lambda x_0, x_1, \bar{x}.b^\sim} \quad \frac{\Gamma \vdash f : \Pi A\, B \quad \Gamma \vdash u : A}{\Gamma^= \vdash (f\ u)^\sim \equiv f^\sim\ u[0]\ u[1]\ u^\sim}$$

For types, we choose:

$$\frac{\Gamma \vdash A : \mathrm{U}}{\Gamma^= \vdash A^\sim \equiv \sim_A}$$

Homogeneous equality

To internalise the logical relation, i.e. to have an equality in the same context, we define the substitution R and the term refl mutually:

$$\frac{\vdash \Gamma}{\mathsf{R}_\Gamma : \Gamma \Rightarrow \Gamma^=} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}\, a \equiv (a^\sim)[\mathsf{R}_\Gamma] : a \sim_A [\mathsf{R}_\Gamma]\, a}$$

$$\emptyset \vdash \mathsf{R}_\emptyset \equiv () : \emptyset$$
$$\Gamma.x : A \vdash \mathsf{R}_{\Gamma.A} \equiv (\mathsf{R}_\Gamma, x, x, \mathsf{refl}\, x) : (\Gamma.A)^=$$

We introduce the abbreviation:

$$a =_A b \equiv a \sim_A [\mathsf{R}]\, b$$

We also need an $\mathsf{S}_\Gamma : (\Gamma^=)^= \Rightarrow (\Gamma^=)^=$, with a family of similar operations to refl.

The functor $-^{=}$ (i)

We can extend $-^{=}$ to act not only on contexts, but also terms, and substitutions:

$$
\begin{aligned}
\Gamma &\mapsto \Gamma^{=} \\
\Gamma \vdash t : A &\mapsto \Gamma^{=} \vdash t^{=} \equiv (t[0], t[1], t^{\sim}) : A^{=} \\
(\rho, x \mapsto t) &\mapsto (\rho^{=}, t^{=})
\end{aligned}
$$

Higher dimensions (i)

By iterating $-^{=}$, we get higher dimensional cubes:

$$
\begin{array}{ll}
\Gamma & \vdash x : A \\
\Gamma^{=} & \vdash (x:A)^1 \equiv \quad x_0 : A[0_\Gamma].x_1 : A[1_\Gamma].\bar{x} : x_0 \sim_A x_1 \\
\Gamma^2 & \vdash (x:A)^2 \equiv
\end{array}
$$

$$
\begin{array}{lll}
x_{00} : A[0_\Gamma 0_{\Gamma^=}] & .x_{01} : A[0_\Gamma 1_{\Gamma^=}] & .\bar{x_0} : x_{00} \sim_{A[0_\Gamma]} x_{01} \\
.x_{10} : A[1_\Gamma 0_{\Gamma^= .A[0]}] & .x_{11} : A[1_\Gamma 1_{\Gamma^= .A[0]}] & .\bar{x_1} : x_{10} \sim_{A[1_\Gamma]} x_{11} \\
.(\bar{x})_0 : (x_0 \sim_A x_1)[0_{\Gamma^= .A[0].A[1]}] & .(\bar{x})_1 : (x_0 \sim_A x_1)[1_{\Gamma^= .A[0].A[1]}] & .\bar{\bar{x}} : (\bar{x})_0 \sim_{x_0 \sim_A x_1} (\bar{x})_1
\end{array}
$$

$$
\Gamma^3 \quad \vdash (x:A)^3 \equiv
$$

$$
\begin{array}{lll}
x_{000} : A[000] & .x_{001} : A[001] & .\bar{x_{00}} : x_{000} \sim_{A[00]} x_{001} \\
.x_{010} : A[010] & .x_{011} : A[011] & .\bar{x_{01}} : x_{010} \sim_{A[01]} x_{011} \\
.(\bar{x_0})_0 : x_{000} \sim_{A[0]} [0]\, x_{010} & .(\bar{x_0})_1 : x_{001} \sim_{A[0]} [1]\, x_{011} & .\bar{\bar{x_0}} : (\bar{x_0})_0 \sim_{x_{00} \sim_{A[0]} x_{01}} (\bar{x_0})_1 \\
.x_{100} : A[100] & .x_{101} : A[101] & .\bar{x_{10}} : x_{100} \sim_{A[10]} x_{101} \\
.x_{110} : A[110] & .x_{111} : A[111] & .\bar{x_{11}} : x_{110} \sim_{A[11]} x_{111} \\
.(\bar{x_1})_0 : x_{100} \sim_{A[1]} [0]\, x_{110} & .(\bar{x_1})_1 : x_{101} \sim_{A[1]} [1]\, x_{111} & .\bar{\bar{x_1}} : (\bar{x_1})_0 \sim_{x_{10} \sim_{A[1]} x_{11}} (\bar{x_1})_1 \\
.(\bar{x})_{00} : x_{000} \sim_A [00]\, x_{100} & .(\bar{x})_{01} : x_{001} \sim_A [01]\, x_{101} & .\overline{(\bar{x})_0} : (\bar{x})_{00} \sim_{x_{00} \sim_{A[0]} x_{10}} (\bar{x})_{01} \\
.(\bar{x})_{10} : x_{010} \sim_A [10]\, x_{110} & .(\bar{x})_{11} : x_{011} \sim_A [11]\, x_{111} & .\overline{(\bar{x})_1} : (\bar{x})_{10} \sim_{x_{01} \sim_{A[1]} x_{11}} (\bar{x})_{11} \\
.(\bar{\bar{x}})_0 : (\bar{x})_{00} \sim_{x_0 \sim_A x_1} [0]\, (\bar{x})_{10} & .(\bar{\bar{x}})_1 : (\bar{x})_{01} \sim_{x_0 \sim_A x_1} [1]\, (\bar{x})_{11} & .\bar{\bar{\bar{x}}} : (\bar{\bar{x}})_0 \sim_{(\bar{x})_0 \sim_{x_0 \sim_A x_1} (\bar{x})_1} (\bar{\bar{x}})_1
\end{array}
$$

Higher dimensions (ii)

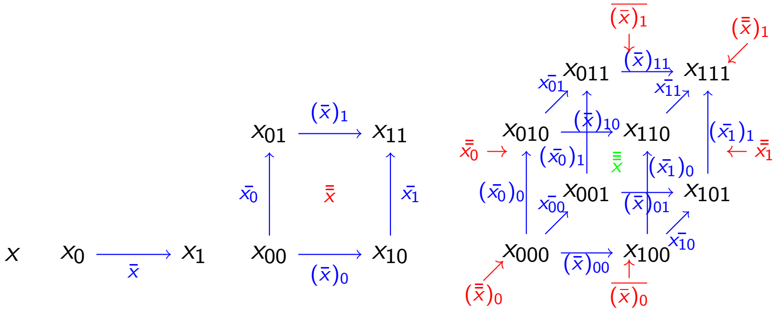


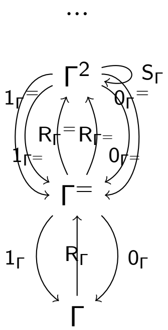
Figure: Cubes of dimension 0-3.

The functor $-^{=}$ (ii)

The iterated version of $-^{=}$ makes any context into a presheaf over the base category of cubical sets.

So, a context Γ is a presheaf $\mathcal{C} \to \mathsf{Con}$ where

- $\mathcal{C}$ is the category of names and substitutions for the cubical set model,
- Con is the category of contexts and substitutions in the term model.



Definition of $\sim_U$

Our previous definition:

$$A \sim_U B \equiv A \to B \to U$$

We replace this by:

$$\begin{aligned}
\Gamma \vdash A \sim_U B \equiv \; \Sigma - \sim - &: A \to B \to U \\
\mathrm{coe}^0 &: A \to B \\
\mathrm{coh}^0 &: \Pi(x : A).x \sim \mathrm{coe}^0\, x \\
\mathrm{uni}^0 &: \Pi(x : A, p\, p' : \Sigma(y : B).x \sim y).p = p' \\
\mathrm{coe}^1 &: B \to A \\
\mathrm{coh}^1 &: \Pi(y : B).\mathrm{coe}^1\, y \sim y \\
\mathrm{uni}^1 &: \Pi(y : B, p\, p' : \Sigma(x : A).x \sim y).p = p'
\end{aligned}$$

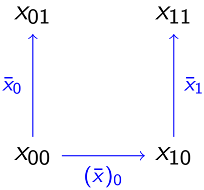
Kan conditions (i)

We are required to provide coe and coh now for each type former.
For Σ we can define it as

$$\begin{aligned}
\Gamma^{=} \vdash \mathsf{coe}^i_{\Sigma(x:A).B} &\equiv \lambda(a,b).(\mathsf{coe}^i_A\, a, \mathsf{coe}^i_B[\bar{x} \mapsto \mathsf{coh}^1_i\, a]\, b) \\
&: (\Sigma(x:A).B)[i] \to (\Sigma(x:A).B)[1-i] \\
\Gamma^{=} \vdash \mathsf{coh}^i_{\Sigma(x:A).B} &\equiv \lambda(a,b).(\mathsf{coh}^i_A\, a, \mathsf{coh}^i_B[\bar{x} \mapsto \mathsf{coh}^1_i\, a]\, b) \\
&: \Pi(w : (\Sigma(x:A).B)[i])\,.\, w \overset{i}{\sim}_{\Sigma(x:A).B} \mathsf{coe}^1_i\, w
\end{aligned}$$

Kan conditions (ii)

coe and coh can be seen as first level Kan operations: given a point, they extend it to a line.
A higher level Kan operation completes an incomplete square, 3-dimensional cube, etc. Eg.:



To define the first level Kan operations for Π, we need the second level Kan operations. However,

$$\Gamma^{=}.x_0 : A[0].x_1 : A[1] \vdash x_0 \sim_A x_1 : \mathsf{U},$$

so

$$(\Gamma^{=}.x_0 : A[0].x_1 : A[1])^{=} \vdash (x_0 \sim_A x_1)^{\sim} : (x_{00} \sim_A [0]\, x_{10}) \sim_{\mathsf{U}} (x_{01} \sim_A [1]\, x_{11}).$$

Kan for Π

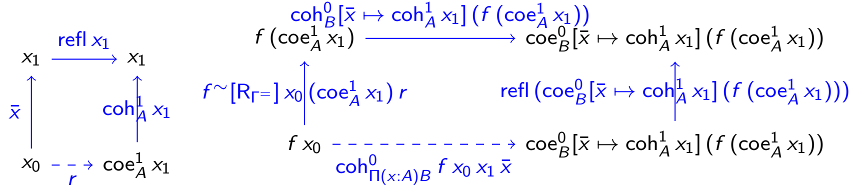
Coerce for Π:

$$\Gamma^{=} \vdash \mathsf{coe}^0_{\Pi(x:A).B} \equiv \lambda f.\lambda x.\mathsf{coe}^0_B[\bar{x} \mapsto \mathsf{coh}^1_A\, x]\,(f\,(\mathsf{coe}^1_A\, x))$$
$$: (\Pi(x : A[0]).B[0, x]) \to (\Pi(x : A[1]).B[1, x])$$

The type of the coherence operation:

$$\Gamma^{=} \vdash \mathsf{coh}^0_{\Pi(x:A).B} : \Pi\,(f : (\Pi(x : A).B)[0].\, f \sim_{\Pi(x:A).B} (\mathsf{coe}^0_{\Pi(x:A).B}\, f)$$

We get coherence by using higher level Kan:



Identity type (i)

Non-dependent eliminator:

$$\frac{\Gamma \vdash P : A \to \mathsf{U} \quad \Gamma \vdash r : x \sim_A [\mathsf{R}_\Gamma]\, y \quad \Gamma \vdash u : P\, x}{\Gamma \vdash \mathsf{transport}_P\, r\, u : P\, y}$$

We have that P is a congruence:

$$\frac{\Gamma \vdash P : A \to \mathsf{U}}{\Gamma \vdash P^{\sim}[\mathsf{R}_\Gamma] : \Pi(x_0, x_1 : A, \bar{x} : x_0 \sim_A [\mathsf{R}_\Gamma]\, x_1).P\, x_0 \sim_{\mathsf{U}} P\, x_1}$$

And we define transport by using $P^{\sim}[\mathsf{R}]$:

$$\frac{\Gamma \vdash P : A \to \mathsf{U} \quad \Gamma \vdash r : x =_A y \quad \Gamma \vdash u : P\, x}{\Gamma \vdash \mathsf{transport}_P\, r\, u \equiv (P^{\sim}[\mathsf{R}_\Gamma]\, x\, y\, r).\mathsf{coe}^0\, u : P\, y}$$

Identity type (ii)

The computation rule of transport says that $\mathsf{transport}_P(\mathsf{refl}\,x) \equiv \mathsf{id}$. We have

$$\begin{aligned} & \mathsf{transport}_P(\mathsf{refl}\,x) \\ \equiv\, & (P^{\sim}[\mathsf{R}_\Gamma]\,x\,x\,x^{\sim}[\mathsf{R}_\Gamma]).\mathsf{coe}^0 \\ \equiv\, & (P\,x)^{\sim}[\mathsf{R}_\Gamma].\mathsf{coe}^0 \\ \equiv\, & \mathsf{id} \end{aligned}$$

The last step is justified by adding the following rule:

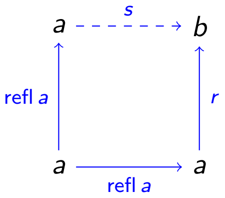
$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma \vdash A^{\sim}[\mathsf{R}_\Gamma] \equiv (- \sim_A [\mathsf{R}_\Gamma] -, \mathsf{id}, \mathsf{refl}, \mathsf{id}, \mathsf{refl}) : A \sim_{\mathsf{U}} A}$$

Identity type (iii)

We also show that singletons are contractible i.e. we show how to construct the terms s and t of the following type:

$$\frac{\Gamma \vdash a, b : A \quad \Gamma \vdash r : a =_A b}{\begin{aligned}\Gamma \vdash (s, t) : \quad & (a, \mathsf{refl}\, a) =_{\Sigma(x:A).a =_A x} (b, r) \\ \equiv \quad & \Sigma(s : a \sim_A [\mathsf{R}_\Gamma]\, b).\mathsf{refl}\, a \sim_{a \sim_A [\mathsf{R}_\Gamma] x} [\mathsf{R}_\Gamma, a, b, s]\, r\end{aligned}}$$

s is constructed by filling the following incomplete square from bottom to top:



Conclusion

- A different presentation of Bernardy and Moulin's work on internal parametricity: equality defined as a logical relation.
- Using equivalence for elements of the universe instead of any relation.
- This forces us to define the first level Kan operations for type formers.
- Higher Kan operations can be computed from the first level Kan operations.
- Unfinished work:
  - Relation to the uniformity condition in the Bezem, Coquand, Huber cubical set model
  - Prove decidability, canonicity
  - Examples of higher inductive types
  - Implement it in Agda
  - Create a proof assistant based on this theory :)